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COMMENTARY

Comment on "A note on the principal measure and distributional (p, q)-chaos of a coupled lattice system related with Belusov–Zhabotinskii reaction"

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Abstract In García Guirao and Lampart (J Math Chem 48:159–164, 2010) presented a lattice dynamical system stated by Kaneko (Phys Rev Lett 65:1391–1394, 1990) which is related to the Belusov–Zhabotinskii reaction. In this note, we give an example which shows that the proofs of Theorems 3.1 and 3.2 in [J Math Chem 51:1410–1417, 2013] are incorrect, and two open problems.

Keywords Coupled map lattice · Distributional (p, q)-chaos · Principal measure · Devaney's chaos · Chaos in the sense of Li–Yorke · Tent map

Mathematics Subject Classification 54H20 · 37B40 · 37D45

1 Introduction

By a topological dynamical system (t.d.s. for short) we mean a pair (X, f) where X is a compact metric space and f is a continuous map of the space X. Since Li and Yorke introduced the term of chaos in 1975 (see [1]), topological dynamical systems were highly discussed and investigated in the literature (see [2,3]) as they are very good examples of problems coming from the theory of topological dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

Coming from physical/chemical engineering applications, such as digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, a very important generalization of classical discrete dynamical systems has recently appeared as an interesting subject for investigation, we mean the so called

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lattice dynamical Systems. For the importance of these type of systems we refer the reader to [4].

To analyze and know when one of these type of systems has a complicated dynamics or not by the observation of one topological dynamical property is an important and open problem (see [5]). In the same paper, by using the notion of chaos, they characterized the dynamical complexity of a coupled lattice system stated by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction, and proved that this coupled map lattice (CML) system is chaotic in the sense of both Devaney and Li–Yorke for zero coupling constant. Furthermore, some problems on the dynamics of this system were stated by them for the case of having non-zero coupling constants. Recently, in [7] the authors proved that for any coupling constant $\varepsilon \in (0, 1)$, this system is chaotic in the sense of both Li–Yorke and has positive entropy.

Distributional chaos, introduced by Schweizer and Smítal in [8], is very interesting and important concept, mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to some compact spaces (see [8,9]). We know that this equivalence does not transfer to higher dimensions, e.g. positive topological entropy does not imply distributional chaos in the case of triangular maps of the unit square [10] (the same happens when the dimension is zero [11]). In [12], a distributional chaotic minimal system was given.

More recently, in [13] it was proved that for any coupling constant $\varepsilon \in (0, 1)$ and any pair $0 \le p \le q \le 1$, the following coupled lattice system is distributionally (p, q)-chaotic:

$$x_n^{m+1} = (1-\varepsilon) f\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f\left(x_{n-1}^m\right) + f\left(x_{n+1}^m\right)\right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in (0, 1)$ is coupling constant and f is the unimodal map of I. Moreover, they showed that for any coupling constant $\varepsilon \in (0, 1)$, the principal measure of this system is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ where the tent map Λ defined by $\Lambda(x) = 1 - |1 - 2x|$ for any $x \in [0, 1]$. In this note, we will further study the dynamical properties of the following lattice dynamical system:

$$x_n^{m+1} = (1-\varepsilon) f\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right], \tag{2}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in (0, 1)$ is coupling constant and f is the unimodal selfmap of I. In particular, we give an example which shows that the proofs of Theorems 3.1 and 3.2 in [14] are incorrect, and two open problems.

2 Preliminaries

Firstly we recall some notations and some concepts. Throughout this paper, (X, d) is a compact metric space, (X, f) is a topological dynamical system (t.d.s. for short) on a compact metric space X with a metric d and $I = [0, 1]$.

A pair $x, y \in X$ is a Li–Yorke pair of system (X, f) if and only if the following conditions are satisfied:

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

A subset $S \subset X$ is a LY-scrambled set for f (Li–Yorke set) if and only if the set S has at least two points and every pair of distinct points in S is a Li–Yorke pair. A system (X, f) or a map $f : X \to X$ is chaotic in the sense of Li–Yorke if and only if it has an uncountable scrambled set.

For a given t.d.s. (X, f), any pair of points $x, y \in X$ and a given $n \in \mathbb{N}$, the distributional function $F^n_{xy} : \mathbb{R}^+ \to [0, 1]$ is defined by

$$F^n_{xy}(t) = \frac{1}{n}\sharp\left\{i \in \mathbb{N} : d\left(f^i(x), f^i(y)\right) < t, 1 \le i \le n\right\},$$

where $\mathbb{R}^+ = [0, +\infty)$ and $\sharp$ means the cardinality. Let

$$F_{xy}(t, f) = \liminf_{n\to\infty} F^n_{xy}(t)$$

and

$$F^*_{xy}(t, f) = \limsup_{n\to\infty} F^n_{xy}(t).$$

For any given $0 \le p \le q \le 1$, a t.d.s. (X, f) is distributionally (p, q)-chaotic if and only if there exist an uncountable subset $S \subset X$ and $\varepsilon > 0$ such that $F_{xy}(t, f) = p$ and $F^*_{xy}(t, f) = q$ for any pair of distinct points $x, y \in S$ and any $t \in (0, \varepsilon)$. Particularly, (X, f) is distributionally chaotic if it is distributionally $(0, 1)$-chaotic (see [13–15]).

The principal measure $\mu_p(f)$ of a t.d.s. (X, f) is defined as

$$\mu_p(f) = \sup_{x,y\in X} \frac{1}{D} \int_0^{+\infty} \left(F^*_{xy}(t, f) - F_{xy}(t, f)\right) dt$$

where D =diam (X) is the diameter of the space X (see [16]). From [16] we know that

$$\mu_p(\Lambda) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n + 1)(2^{n-1} + 1)}.$$

The state space of lattice dynamical system (LDS) is the set

$$\mathcal{X} = \left\{x : x = \{x_i\}, x_i \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|x_i\| < \infty\right\}.$$

where $a \ge 1$ is the dimension of the range space of the map of state x_i, $b \ge 1$ is the dimension of the lattice and the l^2 norm

$$\|x\|_2 = \left(\sum_{i \in \mathbb{Z}^b} |x_i|^2 \right)^{\frac{1}{2}}$$

is usually taken ($|x_i|$ is the length of the vector x_i) (see [5]).

We will further consider and investigate the following Coupled Map Lattice system stated by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction (for this point we refer the reader to [17], and for experimental study of chemical turbulence by this method we refer the reader to [18–20]):

$$x_n^{m+1} = (1-\varepsilon) f\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right], \tag{3}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in (0, 1)$ is coupling constant and f is the unimodal selfmap of I.

In general, one of the following periodic boundary conditions of the system (3) is assumed:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

3 Main results

The system (3) was studied by many authors, mostly experimentally or semi-analytically than analytically. The first paper with analytic results is [21], where they showed that this system is Li–Yorke chaotic. In [5] the authors presented a different and easier proof of this result.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L)) = \left(\sum_{i=1}^{L} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L) \in I^L$.

Define a map $F : (I^L, d) \to (I^L, d)$ by $F(x_1, x_2, \ldots, x_L) = (y_1, y_2, \ldots, y_L)$ where $y_i = (1-\varepsilon) f(x_i) + \frac{\varepsilon}{2}(f(x_{i-1}) - f(x_{i+1}))$. It is clear to see that the system (3) is equivalent to the above system (I^L, F). Note that the system (3) is different from the system (1) when $\varepsilon \neq 0$. In [5] the authors pointed out that for non-zero couplings constants, this lattice dynamical system (3) is more complicated.

Example 3.1 If $\varepsilon = \frac{1}{2}$ and $x_0 = \frac{1}{4}$, then

$$F^n((x_0, x_0, \ldots, x_0)) \neq (1-\varepsilon)(\Lambda^n(x_0), \Lambda^n(x_0), \ldots, \Lambda^n(x_0))$$

for any $n \geq 2$. In fact, if $\varepsilon = \frac{1}{2}$ and $x_0 = \frac{1}{4}$, then we have that

$$F^n((x_0, x_0, \ldots, x_0)) = \left(\frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}\right)$$

for any $n \geq 1$, and that

$$\left(1 - \frac{1}{2}\right)(\Lambda(x_0), \Lambda(x_0), \ldots, \Lambda(x_0)) = \left(\frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}\right),$$

$$\left(1 - \frac{1}{2}\right)(\Lambda^2(x_0), \Lambda^2(x_0), \ldots, \Lambda^2(x_0)) = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$$

and

$$x\left(1 - \frac{1}{2}\right)(\Lambda^n(x_0), \Lambda^n(x_0), \ldots, \Lambda^n(x_0)) = (0, 0, \ldots, 0)$$

for any $n \geq 3$. Consequently,

$$F^n((x_0, x_0, \ldots, x_0)) \neq (1 - \varepsilon)(\Lambda^n(x_0), \Lambda^n(x_0), \ldots, \Lambda^n(x_0))$$

for any $n \geq 2$.

Remark 3.1 Example 3.1 shows that the proofs of Theorems 3.1 and 3.2 in [14] are wrong.

Problem 3.1 Let $f = \Lambda$, $0 \leq p \leq q \leq 1$ and $\varepsilon \in (0, 1)$. Is the system (3) distributionally (p, q)-chaotic?

Problem 3.2 Let $\varepsilon \in (0, 1)$. Is the principal measure of the system (3) not less than $\mu_p((1 - \varepsilon) f)$?

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